Math 100 - Week 3 Recitation (Fall 2013)

So far we've developed two major techniques for taking antiderivatives: u-substitution and integration by parts. For the integrals in Problems 1 through 10, your goal is to determine whether solving the integral would require:

(A) Substitution	(C) Both of these methods
(B) Integration by parts	(D) Neither of these methods

Note that figuring out which method to use might not require you to completely solve the integral. You should start by discussing with your group what strategy you would use for each integral, and then "check your work" by solving as many integrals as you have time for. You should probably prioritize the integrals you are the least sure about! (Plus constants of integration!)

1. B

$$\int x^5 e^x \, dx$$

Let $u = x^5 \Rightarrow du = 5x^4$, $dv = e^x dx \Rightarrow v = e^x$.

$\Rightarrow\ = e^x \cdot x^5 - 5\int x^4 e^x dx$ (Now another Integration by Parts)

$= e^x x^5 - 5e^x \cdot x^4 + \int 20x^3 e^x dx = \dots$ [Continue]

$= e^x(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120)$

2. A

$$\int x^5 \sin(x^6) \, dx$$

Let $u = x^6$. $du = 6x^5 dx$ $\Rightarrow\ = \frac{1}{6}\int \sin u \, du = -\frac{1}{6}\cos u = -\frac{1}{6}\cos x^6$

3. A

$$\int x^3 e^{(x^4)} \, dx$$

Let $u = x^4$. $du = 4x^3 dx$ $\Rightarrow\ = \frac{1}{4}\int e^u du = \frac{1}{4}e^u = \frac{1}{4}e^{x^4}$

4. C

$$\int x^9 \sin(x^5) \, dx$$

Let $u = x^5$. $du = 5x^4 dx$

$\Rightarrow\ = \frac{1}{5}\int x^5 \sin u \, du = \frac{1}{5}\int u \sin u \, du.$

Now Integrate by parts:

$= \frac{1}{5}\left(-u\cos u - \int -\cos u \, du\right) = \frac{1}{5}(\sin u - u\cos u) = \frac{1}{5}(\sin x^5 - x^5 \cos x^5)$

5. D

$$\int (x^5 + 1)(x^6 + 1) \, dx$$

Distribute the Factors:

$= \int x^{11} + x^6 + x^5 + 1 \, dx$

$= \frac{1}{12}x^{12} + \frac{1}{7}x^7 + \frac{1}{6}x^6 + x$

6. A

$$\int x^5 \sqrt{x^6 + 1} \, dx$$

Let $u = x^6 + 1$. $du = 6x^5 dx$ $\Rightarrow$

$= \frac{1}{6}\int \sqrt{u} \, du = \frac{1}{6} \cdot \frac{2}{3}u^{3/2} = \frac{u^{3/2}}{9} = \frac{(x^6+1)^{3/2}}{9}$

7. B

$$\int (x^5 + x^2) \ln x \, dx$$

Let $u = \log x$, $du = \frac{1}{x}dx$, $dv = (x^5 + x^2)dx$, $v = \frac{1}{6}x^6 + \frac{1}{3}x^3$

$= \left(\frac{1}{6}x^6 + \frac{1}{3}x^3\right)\log x - \int\left(\frac{1}{6}x^6 + \frac{1}{3}x^3\right)\frac{dx}{x}$

$= \left(\frac{1}{6}x^6 + \frac{1}{3}x^3\right)\log x - \int \frac{1}{6}x^5 + \frac{1}{3}x^2 \, dx$

$= \left(\frac{1}{6}x^6 + \frac{1}{3}x^3\right)\log x - \frac{x^6}{36} - \frac{x^3}{9}.$

8. D

$$\int \frac{1}{\sqrt{1 - x^2}} \, dx$$

Remember $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \Rightarrow \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x.$

9. A

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx$$

Let $u = \sqrt{x}$, $du = \frac{1}{2}x^{-1/2}dx$ $\Rightarrow$

$= 2\int e^u du = 2e^u = 2e^{\sqrt{x}}.$

10. C

$$\int e^{\sqrt{x}} \, dx$$

Let $u = \sqrt{x}$. $du = \frac{1}{2}x^{-1/2}dx$ $\Rightarrow dx = 2\sqrt{x}\,du$

$\sqrt{x} = u$ $\quad t = u.$

$\Rightarrow\ = \int e^u 2\sqrt{x}\,du = 2\int e^u u \, du = 2\int e^t t \, dt.$

Now let $u = t$, $dv = e^t dt$ $\Rightarrow du = dt$, $v = e^t$.

$= 2\left(te^t - \int e^t dt\right) = 2te^t - 2e^t = 2e^{\sqrt{x}}(\sqrt{x} - 1).$

11. (Challenge problem!) Suppose that after t seconds, starting at time $t = 1$, an object is $\sin(\ln t)$ meters above the ground. Note that the object is on the ground at time $t = 1$.

(a) At what time will the object first hit the ground again?

(b) Find the average height of the object during the time between when the object leaves the ground ($t = 1$) and when it first hits the ground again.

(a) The object hits the ground again the first time $\sin(\log t) = 0$, for $t \geq 1$. Since $\log(t)$ is increasing, This first 0 happens when $\log t = z$, where z is the smallest positive root of $\sin x$. So $z = \pi$, and $\boxed{t = e^{\pi}.}$

(b) Average height is given by the average value of the height function over the interval $[1, e^{\pi}]$. So Average height $= AH = \frac{1}{e^{\pi} - 1}\int_1^{e^{\pi}} \sin(\log t)\,dt$. To solve this, let $x = \log t$. Then $dx = \frac{dt}{t}$, so $dt = t\,dx = e^x dx$.

$\Rightarrow AH = \frac{1}{e^{\pi}-1}\int_0^{\pi} \sin x \, e^x dx$. Now $\int \sin x \, e^x dx = \sin x \, e^x - \int \cos x \, e^x dx = \sin x \, e^x - \cos x \, e^x + \int \sin x \, e^x dx$ by Integration by parts (twice), so $\int_0^{\pi} \sin x \, e^x dx = \frac{1}{2}e^x(\sin x - \cos x)\Big|_0^{\pi} = \frac{1}{2}(e^{\pi} + 1)$. Then $AH = \boxed{\frac{1}{2} \cdot \frac{e^{\pi}+1}{e^{\pi}-1}} \approx 0.545166.$

Note: the answer can also be written as $\frac{1}{2}\coth\left(\frac{\pi}{2}\right)$, with coth the hyperbolic cotangent.