TIONTOBL

MICHELLE DELCOURT, DAVID HOLLIS, AND DAVID LOWRY

Abstract. In this paper we propose a combinatorial two player game, “Tiontobl”, which is played on the diagonals of a convex n-gon. Any game of Tiontobl can be simplified down into a game of Snort (invented by S. Norton) through a process dubbed “Snortification”. However, a game of Snort cannot be uniquely converted into a game of Tiontobl. We prove that a game of Tiontobl is a first player win for $n > 3$.

1. The Game Tiontobl

1.1. How to Play. Tiontobl (pronounced shun-*tah*-bl) is a game between two players played on a convex n-gon in a plane. We adopt Berlekamp's convention[1] of referring to the players as Left and Right (colored bLue and Red, respectively). The players take turns drawing diagonals of the polygon, such that

- each diagonal connects two nonadjacent vertices, and
- no two diagonals of different colors intersect.

The first player who, at the start of his or her turn, has no legal move loses.

1.2. Tiontobl is Combinatorial.

Definition 1. We will use the symbol T_n to denote the blank Tiontobl game on n vertices.

A game of Tiontobl on n vertices is

- **finite:** There are at most $\frac{n(n-3)}{2}$ diagonals uncolored, and no diagonal can be colored twice. Thus, the game T_n terminates in at most $\frac{n(n-3)}{2}$ moves.
- **deterministic:** Each player may choose any valid move. There are no random moves.

2. Analysis of Simple Games

We can analyze the games T_n with $n < 6$ fairly simply by hand:

- $n = 3$: The game T_3 has value zero; since triangles have no interior diagonals, neither player has a legal first move.
- $n = 4$: T_4 is ± 1, for if Left moves first, Right has no legal moves, and vice versa.
- $n = 5$: Play in T_5 will always terminate in exactly 3 turns; each player has only one choice, up to symmetry. $T_5 = *$.

T_6, however, is more complex. We need to consider using more sophisticated methods to analyze larger games.

3. Even Games

For games T_n where n is even, there is a simple strategy for the first player.

Lemma 1. *T_n is a first player win for $n > 2$, n even.*

Proof. Without loss of generality, suppose Left moves first. Starting with an arbitrary vertex, label each of the vertices with integers 1 to n clockwise around the board. Left should color bLue the diagonal connecting vertex n and vertex $\frac{n}{2}$. This divides the board into two congruent halves. Every time Right makes a move on one side, the corresponding move will be a legal move for Left on the opposite side. Thus, if Right has a move, Left always has a move following it, and Right will ultimately lose. ∎

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4. Snortification

4.1. **The Game Snort.** Snort is a graph-coloring game presented by John Conway in [2], p.91, and attributed to his colleague, Simon Norton. It is played on a graph whose vertices are initially uncolored. Players Left and Right, in turn, color a single uncolored vertex such that no two adjacent vertices are colored differently. Vertices colored by Left are solid bLue; vertices colored by Right are solid Red. For convenience, while playing, vertices which can only be colored by one player are outlined with that player's color (e.g., a vertex which only Right can color will be outlined in Red), and vertices which neither player can color are crossed out in purple. When, at the beginning of his or her turn, a player cannot color any vertex legally, that player loses. In general, each player wants to maximize potential moves for themselves, while minimizing the number of potential moves for their opponent. More formal strategies for playing games of Snort are discussed in [1] and [2].

4.2. **Snortifying Tiontobl.** Every game G of Tiontobl can be converted into a game of Snort. We call this process *Snortification* and denote the resulting Snort game (called the Snortificate of G) by $\mathrm{Sn}(G)$. We define the vertex set of $\mathrm{Sn}(G)$ to be the set of diagonals within G. bLue diagonals in G correspond to bLue vertices in $\mathrm{Sn}(G)$, and Red diagonals to Red vertices. In $\mathrm{Sn}(G)$, there is an edge between vertices exactly when their corresponding diagonals intersect.

With $\mathrm{Sn}(G)$ so constructed, coloring a diagonal in G corresponds to coloring a vertex in $\mathrm{Sn}(G)$. Any valid move in $\mathrm{Sn}(G)$ is a valid move in G, and vice versa. Thus, if we know an optimal strategy for the Snort game $\mathrm{Sn}(G)$, then we know how to play the Tiontobl game G optimally. A figure showing a game of Tiontobl beginning with T_6, along with a Snort game beginning with $\mathrm{Sn}(T_6)$ can be found in the Appendix.

Lemma 2. *T_n is a first player win for $n > 3$, n odd.*

Proof. Let $n = 2k+1$ for some integer k. Without loss of generality, suppose Left moves first. As before, choose an arbitrary starting vertex and label all of the vertices 1 to $2k+1$, moving clockwise around the board. Left should color the diagonal connecting the vertices labelled $2k+1$ and $k+1$. Now we examine the Snortificate of the resulting game, and from this we find that Right's optimal move is to choose the diagonal between $2k+1$ and k (the "longest" diagonal incident to Left's). From here, Left colors the diagonal from $2k$ to k, leaving Right the free move of coloring the diagonal from $2k$ to $k+1$. These four moves divide the board into two smaller Tiontobl games, T_k and T_{k+1}.

Now we have reduced one game of Tiontobl to a collection of games, and we must address how Left will play such a collection. Left should initiate play in the largest subgame, T_m. If m is even, Left should play T_m as in Lemma 1, until Right has no available moves in T_m. By parity, Left has the next available move, and should once again initiate play in the largest blank subgame. If m is odd, Left should open T_m as above, dividing it into two smaller, independent subgames. Again by parity, it is now Left's turn, and he or she should initiate play in the largest available blank subgame. Since Tiontobl is a combinatorial game, eventually this process must terminate, leaving Right with no available moves and Left to play. At this point, if it is Left's turn (this happens exactly when the last even game is not T_4), Left should simply finish any one of the even subgames, forcing Right to lose the game. □

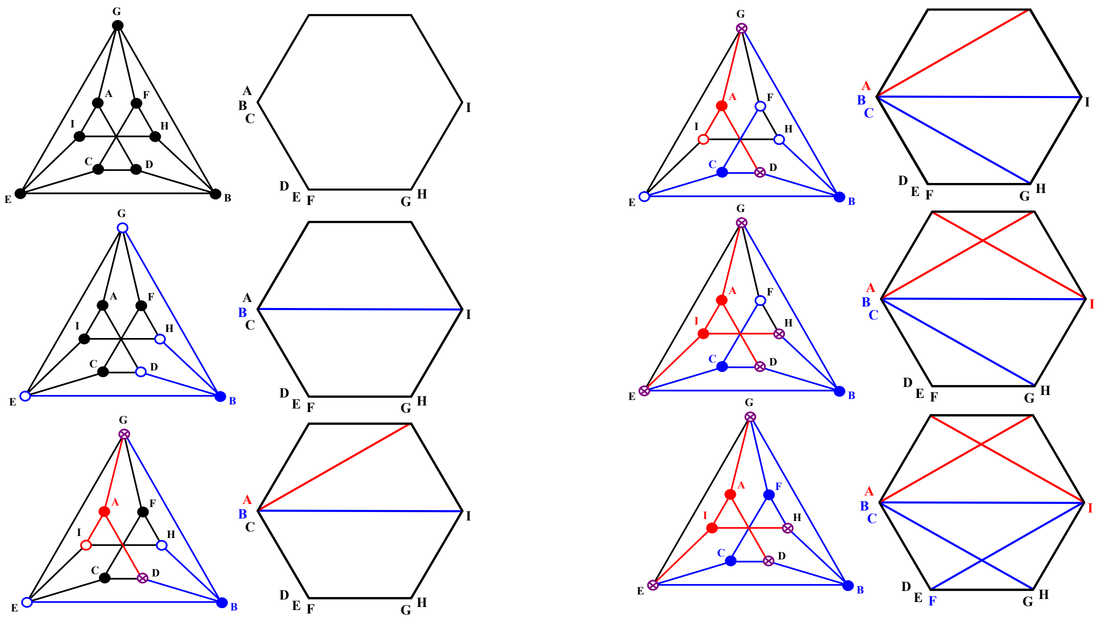
5. Summary of Results

Let G be a blank Tiontobl game, and suppose the best opening move for Left moves the game to G'. Then the best opening move for Right in G is the corresponding move, which moves the game to $-G'$. So, $G = \{G' | -G'\} = \pm G$. So, if $G' < 0$, then $G = 0$; and if $G' \geq 0$, then $G || 0$.

Theorem 1. *$G' = 0$ for $n > 3$ and therefore, T_n is a first-player win.*

Proof. This follows from Lemmas 1 and 2. □

Appendix: Sn(T_6) and T_6, First Player Left (bLue)



References

[1] E.R.Berlekamp, J.H.Conway, R.K.Guy, *Winning ways for your mathematical plays*, Volume 1, A K Peters, Ltd., (2001), pp. 145–188.

[2] J.H.Conway, *On Numbers and Games*, New York Academic Press. (1976), pp. 91–93,96.

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30318
E-mail address: mdelcourt3@gatech.edu

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30318
E-mail address: dhollis6@gatech.edu

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30318
E-mail address: dlowry3@math.gatech.edu